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Technical Memorandum

GENERATION OF DOLPH-CHEBYSHEV WEIGHTS FOR LARGE NUMBERS OF ELEMENTS VIA A FAST FOURIER TRANSFORM

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ABSTRACT

Dolph-Chebyshev weights, which realize a minimum sidelobe level for a specified mainlobe width, can be generated by means of a single Fast Fourier Transform (FFT). For an even number of elements, 2H, the size of the FFT is H. Programs for single precision and double precision procedures are furnished. The utility of this technique for spectral analysis via large-size FFTs is indicated.

ADMINISTRATIVE INFORMATION

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INTRODUCTION

The ability to realize small sidelobes for a weighted equi-spaced line array and specified mainlobe width has been available since Dolph (Ref. 1) solved this problem in 1946. Then in 1953, Stegen (Ref. 2) presented a practical method of obtaining the actual weights to be used for the array. Here we will show that the calculation of all the weights can be accomplished by a single Fast Fourier Transform (FFT). The speed and accuracy of the FFT can thus be used to reduce this once tedious calculation to a very quick and accurate calculation with minimum storage requirements. This is particularly useful for large numbers of elements, such as are encountered in spectral analysis, where the sidelobe problem (equivalent to the array problem) is that of spurious response to frequencies removed from those of interest.

THEORETICAL BACKGROUND

In this section, we shall, for completeness, present the theoretical basis of Dolph's solution and the method Stegen used to calculate the weights. Then we shall show how the weight calculation can be realized by means of a single FFT.

Desireable Pattern

The mathematical problem can be envisioned as selection of the weights $\{w_k\}$ in the finite impulse train in figure 1, such that

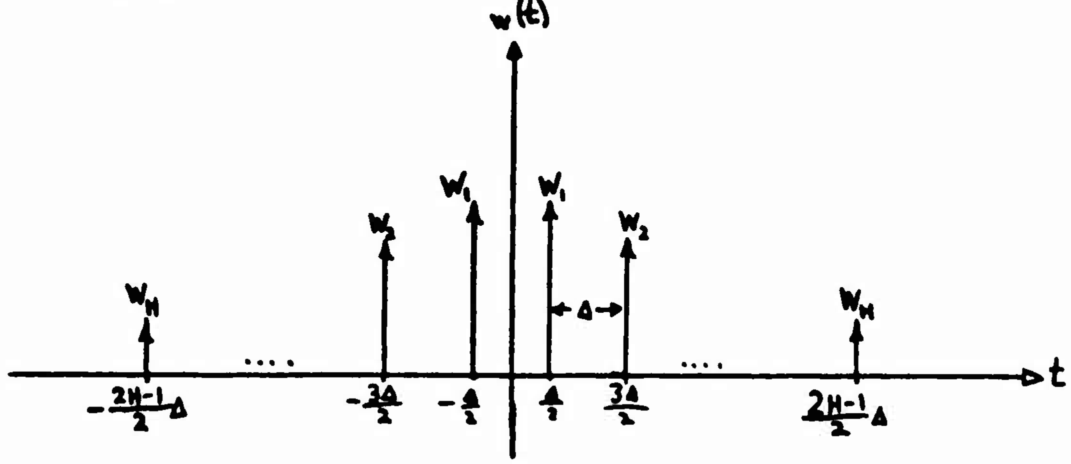

FIGURE 1. Impulse Train for 2H Elements

weight function w(t) has small sidelobes in its spectral window $W(f)$, where

$$W(f) = \int dt\ w(t) \exp(-i2\pi ft) \tag{1}$$

is the Fourier transform of w(t). The total number of elements (impulses) in figure 1 is even, equal to 2H; however, the extension to an odd number of elements is similar to the technique presented in the following. The weights $\{w_k\}$ are assumed to be real and symmetric about the center, t = 0. The time separations between impulse locations are all equal to Δ . Although the problem has been couched in the time domain, the abscissa in figure 1 could equally well be interpreted as distance for the array problem, or as a different variable for other applications.

The weight function w(t) can be expressed as

$$w(t) = \sum_k w_k \,\delta\left(t - \frac{2k-1}{2}\Delta\right). \tag{2}$$

Substitution of (2) into (1) yields

$$\begin{aligned} W(f) &= \sum_k w_k \exp\left[-i2\pi f \frac{2k-1}{2}\Delta\right] \\ &= 2\sum_{k=1}^{H} w_k \cos\left[(2k-1)\pi f\Delta\right], \end{aligned} \tag{3}$$

where we have used the symmetry of the weight structure about the center. The problem now is to choose weights $\{w_k\}_1^H$ such that $W(f)$ in (3) has as low sidelobes as possible for a specified main lobe width. We notice first from (3) that

$$W\left(\frac{1}{\Delta}-f\right) = -W(f) \quad \text{for any } \{w_k\}. \tag{4}$$

Thus only the sidelobes in the region $(0,\frac{1}{2\Delta})$ need be controlled.

Let us define the function

$$G(u) \equiv 2\sum_{k=1}^{H} w_k \cos\left[(2k-1)u\right]; \tag{5}$$

then

$$W(f) = G(\pi\Delta f). \tag{6}$$

The function G satisfies the rules

$$G(-u) = G(u), \quad G(u+\pi) = -G(u), \quad G(\pi - u) = -G(u). \tag{7}$$

Thus, knowledge of $G(u)$ in the interval $(0, \pi/2)$ completely characterizes $G(u)$ for all u; this is consistent with (4) and (6).

As a special case,

$$G(u) = \frac{\sin(2Hu)}{2\sin(u)} \qquad \text{if } w_k = 1 \text{ for } k = 1, 2, \ldots, H. \tag{8}$$

The first null of this function is at $u = \pi/(2H)$; this flat weighting serves as a comparison case.

Now let us consider the function $T_{2H-1}(z_o \cos u)$, where (Ref. 3, 22.3.15)

$$T_{2H-1}(x) = \cos\left[(2H-1) \ \text{arc}\cos(x)\right] \tag{9}$$

is an odd polynomial of degree 2H-1 in x(Ref. 3, 22.3.6). Thus $T_{2H-1}(z_o \cos u)$ is an odd polynomial of degree 2H-1 in cos u and can therefore be expressed as

$$T_{2H-1}(z_o \cos u) = \sum_{k=1}^{H} c_k \cos\left[(2k-1)u\right]. \tag{10}$$

A plot of (10) is given in figure 2. It is seen to possess a single

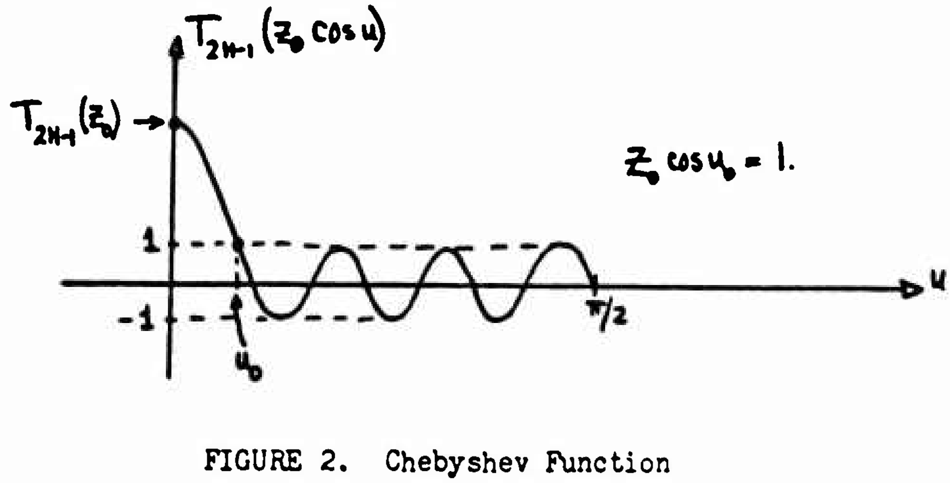

FIGURE 2. Chebyshev Function

large peak (for $z_0 > 1$) at $u = 0$ of value $T_{2H-1}(z_0)$ and equal-ripple sidelobes of value ± 1. The parameter z_0 controls the mainlobe/sidelobe ratio. For large H, (10) can be shown to approach $\cos(\sqrt{A^2u^2-B^2})$ (Ref. 4).

✱ A comparison of (5) and (10) reveals that we can force these two functions to be equal for all u if we please, by proper choice of weights $\{w_k\}$. In so doing, we will realize the highly desirable behavior depicted in figure 2, namely equi-ripple sidelobes and a sharp main lobe. In fact, it was proven by Dolph (Ref. 1) that for a specified mainlobe width (to the first null), the function in figure 2 realizes the minimum sidelobe level possible, and vice versa; thus (10) represents the optimum pattern under this definition.

Determination of Weights

Setting (5) and (10) equal, we require

$$2\sum_{k=1}^{H} w_k \cos\left[(2k-1)u\right] = T_{2H-1}(z_0 \cos u), \text{ all } u. \tag{11}$$

In particular, in order to determine the H unknowns $\{w_k\}_1^H$, let us force equality at

$$u_n = \frac{\pi}{2}\frac{n}{H}, \; n = 0, 1, \ldots, H-1. \tag{12}$$

($G(u)$ and $T_{2H-1}(z_0 \cos u)$ are already equal at $u = \pi/2$.) Then we have

$$2\sum_{k=1}^{H} w_k \cos\left[(2k-1)\frac{\pi}{2}\frac{n}{H}\right] = T_{2H-1}\left(z_0 \cos\left(\frac{\pi}{2}\frac{n}{H}\right)\right), \; n = 0, 1, \ldots, H-1, \tag{13}$$

which constitutes H linear equations in the H unknowns $\{w_k\}_1^H$.

In order to solve* (13), we multiply both sides by $\epsilon_n \cos\left[(2m-1)\frac{\pi}{2}\frac{n}{H}\right]$ and sum on n, where

$$\epsilon_n \equiv \begin{cases} \frac{1}{2}, & n = 0 \\ 1, & n = 1, 2, \ldots, H-1 \end{cases}. \tag{14}$$

---

*This is an application of orthogonality over a discrete set of points; see, for example, Ref. 5, ch. 6.

There results

$$2\sum_{k=1}^{H} w_k \sum_{n=0}^{H-1} \epsilon_n \cos\left[(2k-1)\frac{\pi}{2}\frac{n}{H}\right] \cos\left[(2m-1)\frac{\pi}{2}\frac{n}{H}\right] = \sum_{n=0}^{H-1} \epsilon_n T_{2H-1}\left(z_0 \cos\left(\frac{\pi}{2}\frac{n}{H}\right)\right) \cos\left[(2m-1)\frac{\pi}{2}\frac{n}{H}\right] \tag{15}$$

$$\text{for } m = 1, 2, \ldots, H.$$

Now we employ the closed form expression

$$\sum_{k=0}^{H} \epsilon_k \cos(k\theta) = \frac{\sin(H\theta)}{2\tan(\theta/2)} \tag{16}$$

(where $\epsilon_H \equiv \frac{1}{2}$) in (15), and obtain

$$w_m = \frac{1}{H}\sum_{n=0}^{H-1} \epsilon_n T_{2H+1}\left(z_0 \cos\left(\frac{\pi}{2}\frac{n}{H}\right)\right) \cos\left[(2m-1)\frac{\pi}{2}\frac{n}{H}\right], \quad m = 1, 2, \ldots, H. \tag{17}$$

This is identical to equation (35) of Ref. 2, except for a scale factor.

<u>Representation as FFT</u>

In Appendix A, it is shown (using (17)) that

$$\left.\begin{aligned} w_{2k} &= Re(\alpha_k) \\ w_{2k-1} &= Re(\alpha_{H+1-k}) \end{aligned}\right\}, \quad 1 \le k \le \left[\frac{H}{2}\right], \tag{18}$$

where

$$\alpha_k \equiv \sum_{n=0}^{H-1} z_n \exp(-i\,2\pi k n/H), \quad \text{all } k, \quad (\text{DFT}) \tag{19}$$

and

$$z_n \equiv \frac{1}{H}\epsilon_n T_{2H-1}\left(z_0 \cos\left(\frac{\pi}{2}\frac{n}{H}\right)\right) \exp\left(i\frac{\pi}{2}\frac{n}{H}\right), \quad n = 0, 1, \ldots, H-1. \tag{20}$$

But $\{\alpha_k\}_0^{H-1}$ in (19) can be computed via an H-point FFT of the sequence $\{z_n\}_0^{H-1}$ in (20). And since $\{\alpha_k\}$ is periodic of period H, (18) can be evaluated from the result of the FFT.

SAMPLE REFERENCE PROGRAMS AND EXAMPLES

It will be noticed from (20) that computation of the H + 1 numbers $\cos\left(\frac{\pi}{2}\frac{n}{H}\right)$, $0 \le n \le H$, suffices to determine all the necessary quantities required for the evaluation of $\{z_n\}_0^{H-1}$, except for the actual evaluation of $T_{2H-1}(\cdot)$. Furthermore, the H-point FFT in (19) needs storage only of the quarter-cosine table $\cos(2\pi\, n/H) = \cos(\frac{\pi}{2} \frac{4n}{H})$, $0 \le n \le \frac{H}{4}$, for the FFTs we employ* (Refs. 6, 7). But these numbers are a subset of those generated above; thus economy of evaluation is realized by saving the appropriate cosine values for the FFT evaluation. This feature is utilized in the programs written.

Two subroutines have been written for the evaluation of the Dolph-Chebyshev weights via (18)-(20). In the first one, DOLPHS, the storage, FFT, and weight evaluation are in single precision. In the second subroutine, DOLPHD, the storage, FFT, and weight evaluation are in double precision. These subroutines use the FFTs in Refs. 6 and 7 respectively (without the quarter-cosine subroutine).* In both subroutines, it was found necessary to evaluate $T_{2H-1}(\cdot)$ via double precision; for example, errors in the third decimal place occurred for H = 512 and -40 dB sidelobes when $T_{2H-1}(\cdot)$ was attempted in single precision.

A sample reference program for the single precision case is listed below:

```
      PARAMETER H=512
      DIMENSION X(H),Y(H),C(H)
      DB=40.
      CALL DOLPHS(H,DB,X,Y,C)
      PRINT 1, C
    1 FORMAT(2X,5E26.8)
      END
```

The variable DB is the desired sidelobe level relative to the peak of the mainlobe, and is in single precision. The single precision weights $\{w_k\}_1^H$ in figure 1 are available in array C as indicated in figure 3; that is, C(1) contains edge element weight w_H, while C(H) contains the weight w_1.

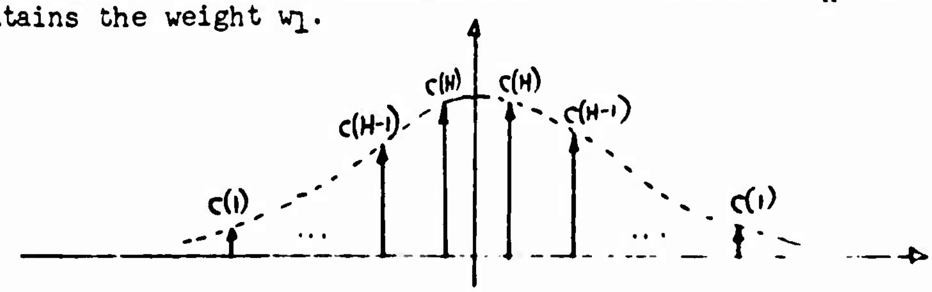

FIGURE 3. Weight Correspondence to C Array

*These FFTs apply only to the case where H is a power of 2; however, the technique of Appendix A is more general.

A sample reference program for the double precision case is listed below:

```
      PARAMETER H=512
      DOUBLE PRECISION X(H),Y(H),C(H)
      DB=40.
      CALL DOLPHD(H,DB,X,Y,C)
      PRINT 1, C
    1 FORMAT(2X,5D26.18)
      END
```

The quantity dB is still in single precision, as above. The double precision weights are available in array C in exactly the same order indicated in figure 3.

For the examples that we tried, the character of the weights $\{w_k\}$ was such that they are always positive, and they decay monotonically from the center, with the exception of the very edge weight, which can be relatively large in some cases of a narrow mainlobe. The spectral response G(u) in (5) is very well approximated by $\cos\left(\sqrt{A^2u^2-B^2}\right)$ for large H (such as 512), as predicted by Ref. 4. The single precision weights agreed with the double precision weights to seven decimals in most cases.* The accuracy of the double precision results were not checked. The time of execution required for CALL DOLPHS and H = 512 was 0.43 seconds on the Univac 1108, Exec 8 system, whereas that for CALL DOLPHD was 0.52 seconds.

The auxiliary arrays X and Y required in the main programs above are available for other purposes once the weights are evaluated and placed in array C; thus the extra storage of 2H cells is a temporary requirement.

Some sample spectral responses are plotted in figures 4-9, normalized to 0 dB at the peak. The first, in figure 4, corresponds to equal weighting at all elements and is presented for comparison purposes. Actually, figure 4 corresponds to a flat continuous weighting over an interval of duration L seconds ($H = \infty$); however, for a large number H of elements, there is no substantial difference between figure 4 and the exact response for finite H. The first null occurs at Lf = 1.

In figure 5, the spectral response for Dolph-Chebyshev weighting with a large number of elements* ($H \gg 1$) is plotted for the case where $Lf = P = 1$ corresponds to the point on the mainlobe where the eventual sidelobe level is first taken on. The mainlobe width (to the first null) is virtully the same as that for figure 4; however, the peak sidelobe has been reduced from -13.3 dB to -21.3 dB.

In figures 6 - 9, the spectral responses for Dolph-Chebyshev weighting are plotted for P = 1.5, 2.0, 2.5, and 3.0 respectively. The quantity P is the value of Lf at which the eventual sidelobe level is first taken on. The broadening of the mainlobe is accompanied by a significant decrease in sidelobe level.

COMMENTS

A quick and accurate technique for evaluating the Dolph-Chebyshev weights for a 2H-element array has been accomplished and employs a single H-point FFT. Thus minimum storage and execution time have been realized; the additional temporary storage needed for the evaluation is available for use immediately after the weights are calculated.
* Double precision evaluation of the Chebyshev polynomial is required; however further calculations can all be done in single precision with excellent results.

The use of Dolph-Chebyshev weighting for spectral analysis via large-size FFTs should be tempered with the following caution: since the spectral response does not decay with frequency (see figures 5-8), strong frequency components far removed from the frequency band of interest can severely bias the estimate. This "distant" bias is greatly reduced when a window (such as Hanning), which decays rapidly with frequency, is used.** On the other hand, the relatively larger adjacent sidelobes of a window with decaying response yield larger values of "local" bias if strong interferences are close to the frequency band of interest. Which window is to be preferred in a particular application depends upon the rate of decay of the true spectrum with frequency, and the strength and location of strong frequency components. Since this information is not known a priori, but is in fact the goal of the analysis, concurrent spectral analysis with two types of windows, e.g. Hanning and Dolph-Chebyshev, may be suggested for some applications.

*The dependence of the spectral response on H is negligible for large H.

**See, for example, Ref. 8, esp. pp. 3 and 10-14.

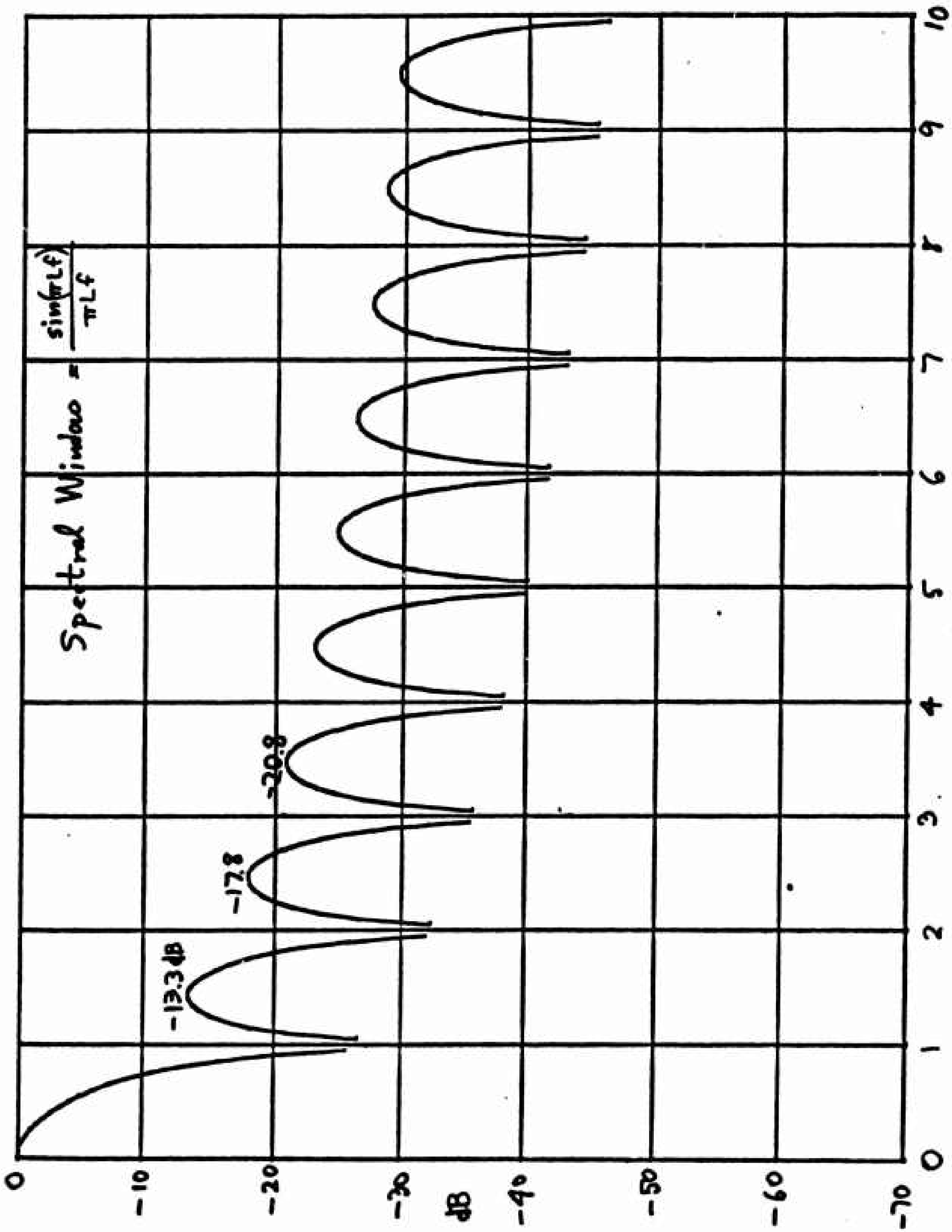

Figure 4. Spectral Response for Flat Time Weighting

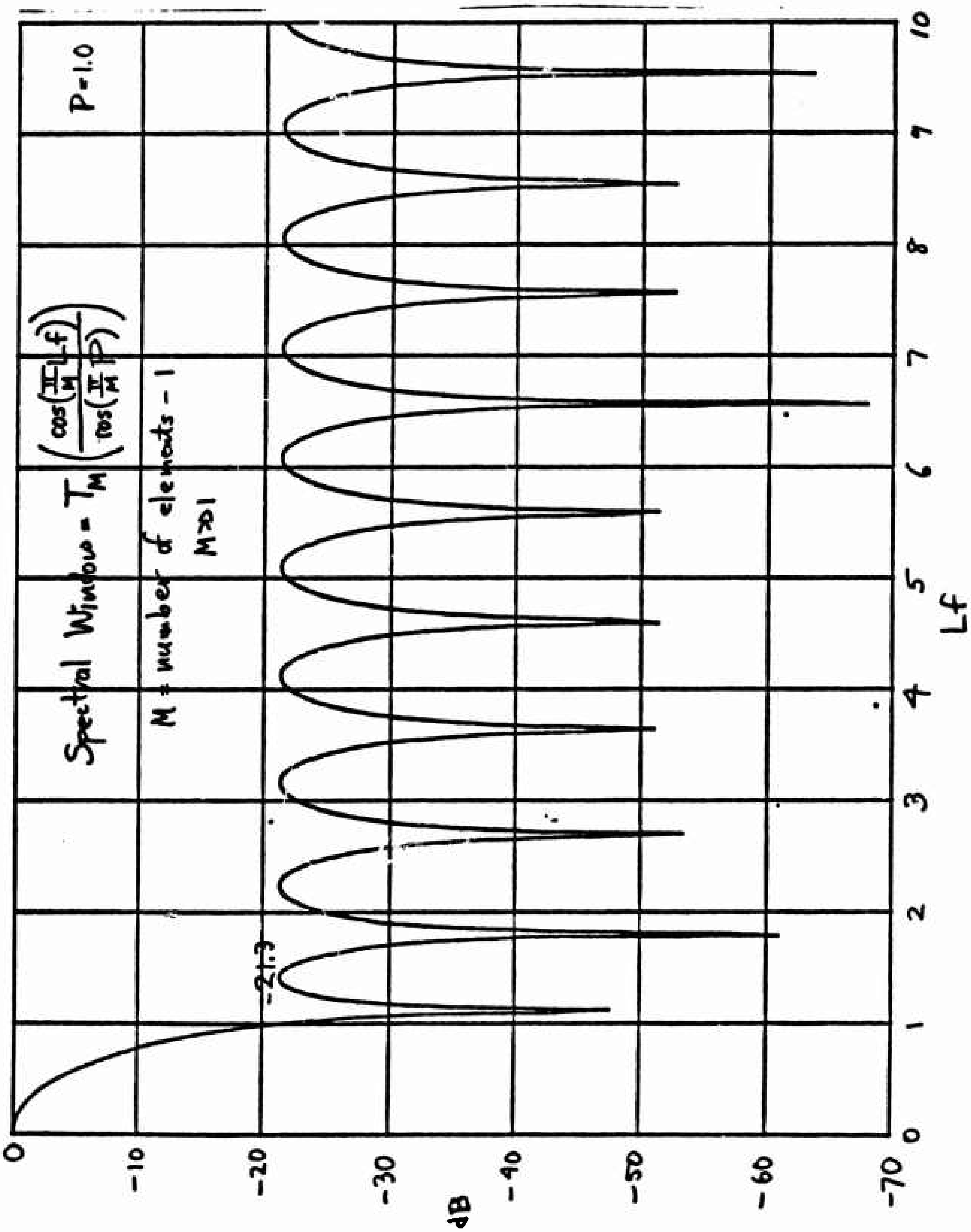

Figure 5. Spectral Response for Dolph-Chebyshev Weighting; P = 1.0

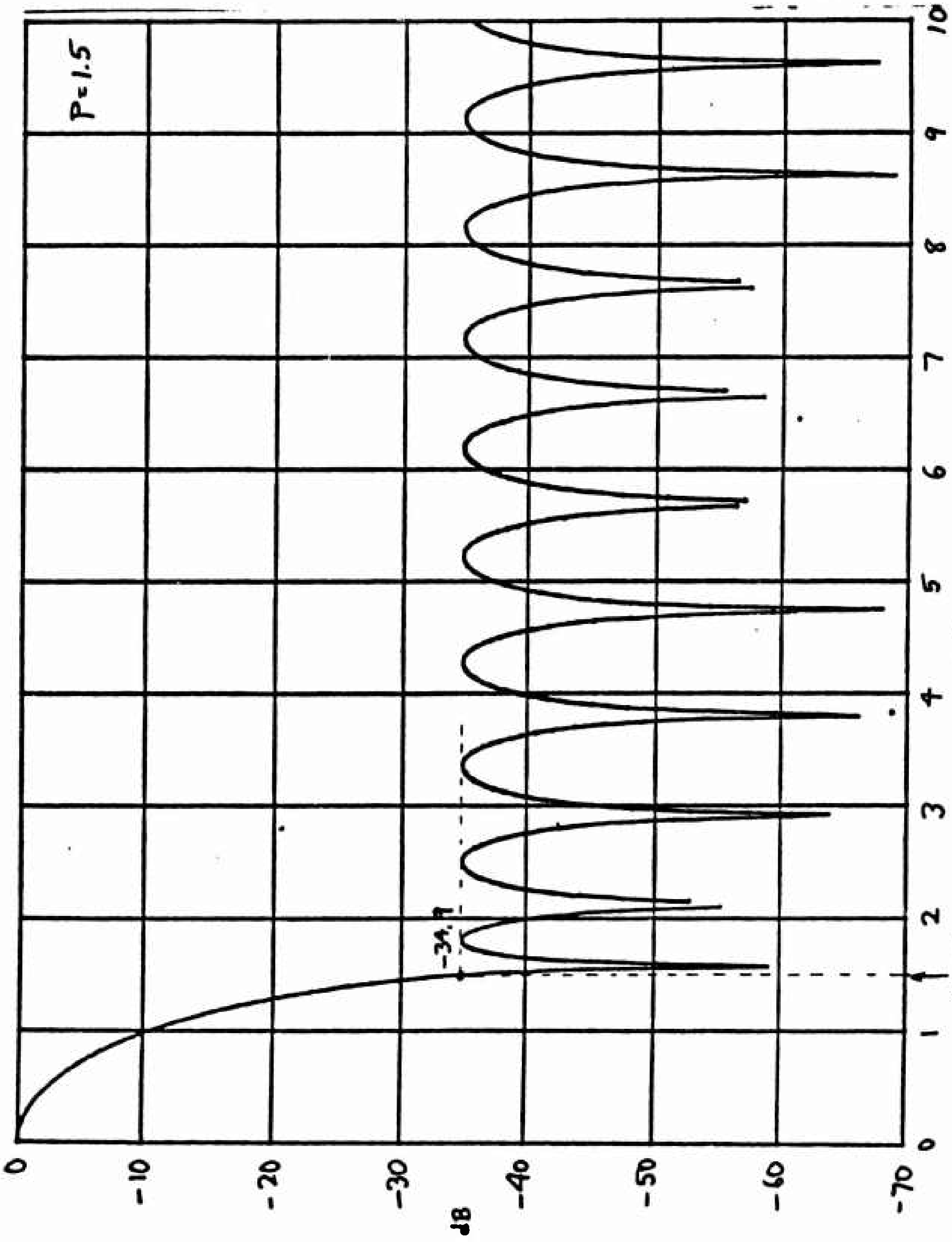

Figure 6. Spectral Response for Dolph-Chebyshev Weighting; P=1.5

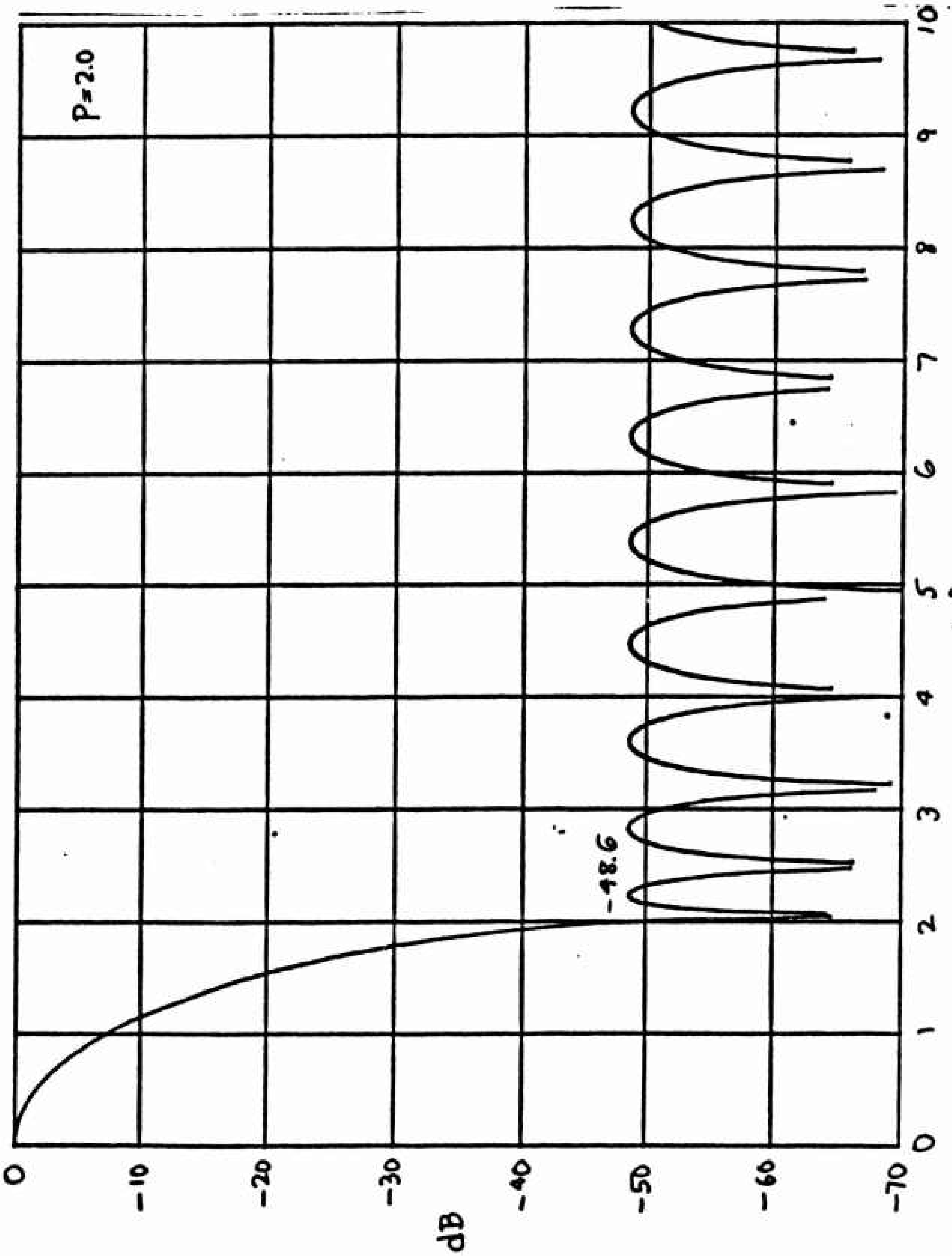

Figure 7. Spectral Response for Dolph-Chebyshev Weighting; P = 2.0

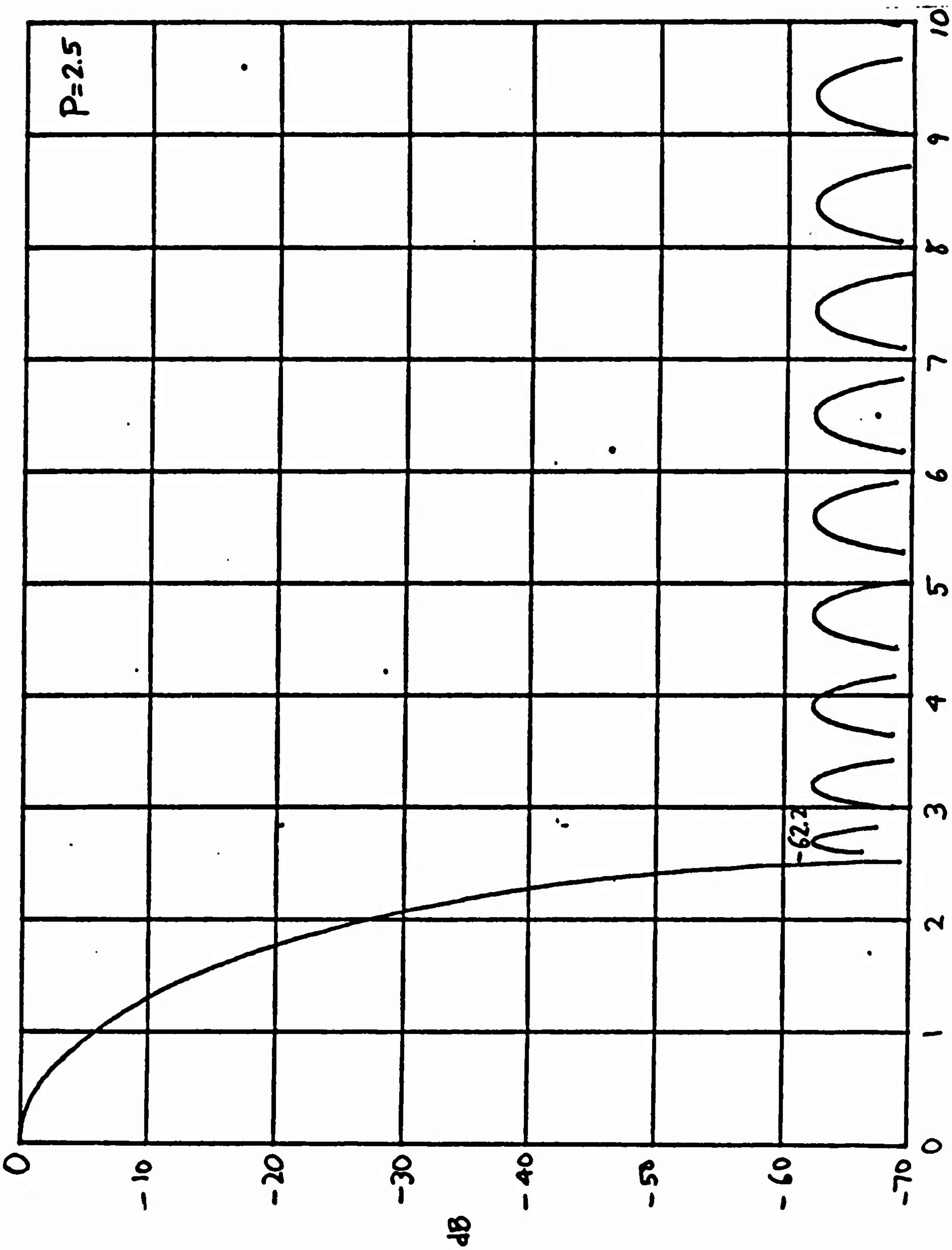

Figure 8. Spectral Response for Dolph-Chebyshev Weighting; P=2.5

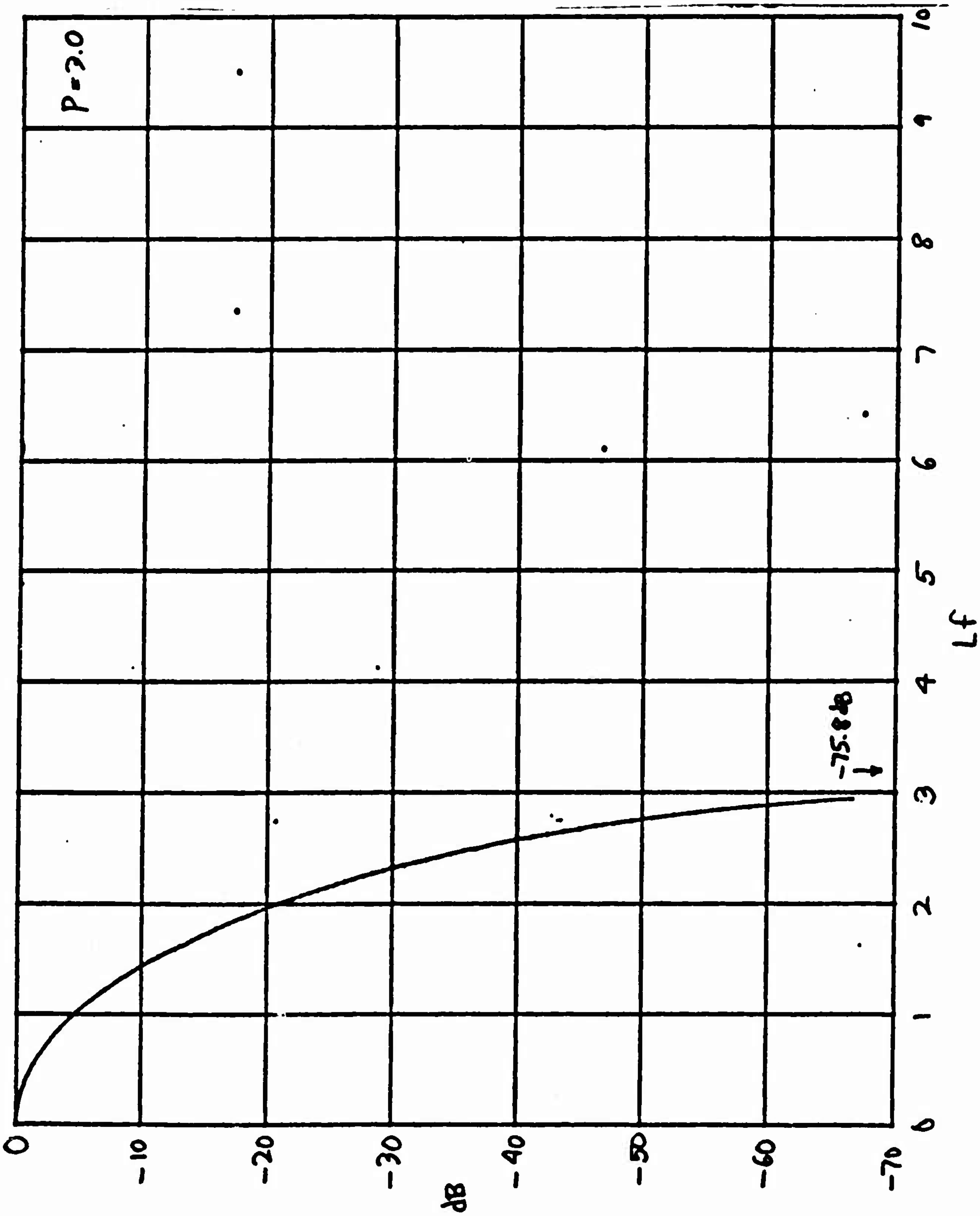

Figure 9. Spectral Response for Dolph-Chebyshev Weighting; P=3.0

REFERENCES

1. C. L. Dolph, "A Current Distribution for Broadside Arrays which Optimizes the Relationship between Beamwidth and Sidelobe Level", Proc. IRE, Vol. 34, pp. 335-348, June 1946.

2. R. J. Stegen, "Excitation Coefficients and Beamwidths of Tschebyscheff Arrays", Proc. IRE, vol. 41, pp. 1671-1674, November 1953.

3. Handbook of Math. Functions, U.S. Dept. of Comm., Nat. Bur. of Stds., AMS No. 55, June 1964.

4. G. J. Van der Maas, "A Simplified Calculation for Dolph-Tchebycheff Arrays", Jour. App. Phys., vol. 25, pp. 121-124, January 1954.

5. R. W. Hamming, Numerical Methods for Scientists and Engineers, McGraw-Hill Book Co., N.Y., 1962.

6. J. F. Ferrie, G. Clifford Carter, and C. W. Nawrocki, "Availability of Markel's FFT Pruning Algorithm", NUSC Tech. Memo. No. TC-1-73, 15 January 1973.

7. J. F. Ferrie, "Double Precision Version of Markel's FFT Algorithm", NUSC Tech. Memo. No. TD113-15-73, 11 June 1973.

8. A. H. Nuttall, "Minimum-Bias Windows for Spectral Estimation by Means of Overlapped Fast Fourier Transform Processing", NUSC Tech. Rpt. 4513, 11 April 1973.

APPENDIX A. MANIPULATION OF (17) INTO FFT FORM

Equation (17) can be expressed as

$$W_m = \mathrm{Re} \sum_{n=0}^{H-1} r_n \exp\left(i\frac{\pi}{2}\frac{n}{H}\right)\exp\left(-i\frac{2\pi m n}{2H}\right), \quad m=1,2,\ldots, H, \tag{A-1}$$

where

$$r_n = \frac{1}{H}\epsilon_n T_{2H-1}\left(z_0 \cos\left(\frac{\pi}{2}\frac{n}{H}\right)\right), \quad n=0,1,\ldots, H-1, \tag{A-2}$$

is a real sequence. Defining

$$z_n = r_n \exp\left(i\frac{\pi}{2}\frac{n}{H}\right), \quad n=0,1,\ldots,H-1, \tag{A-3}$$

(A-1) becomes

$$W_m = \mathrm{Re} \sum_{n=0}^{H-1} z_n \exp\left(-i\frac{2\pi m n}{2H}\right), \quad m=1,2,\ldots,H. \tag{A-4}$$

Now let us define

$$\alpha_k = \sum_{n=0}^{H-1} z_n \exp(-i2\pi k n/H), \quad \text{all } k. \tag{A-5}$$

Then from (A-4) and (A-5),

$$W_{2k} = \mathrm{Re}(\alpha_k), \quad 1 \le 2k \le H. \tag{A-6}$$

Also, from (A-4) and (A-3),

$$\begin{aligned} W_{2k-1} &= \mathrm{Re} \sum_{n=0}^{H-1} z_n \exp\left(i\frac{2\pi n}{2H}\right)\exp(-i2\pi k n/H) \\ &= \mathrm{Re} \sum_{n=0}^{H-1} r_n \exp\left(i\frac{3\pi n}{2H}\right)\exp(-i2\pi k n/H), \quad 1 \le 2k-1 \le H, \end{aligned} \tag{A-7}$$

whereas from (A-5) and (A-3),

$$\begin{aligned} \alpha^*_{H+1-k} &= \sum_{n=0}^{H-1} z_n^* \exp(i2\pi(H+1-k)n/H) \\ &= \sum_{n=0}^{H-1} r_n \exp\left(-i\frac{\pi}{2}\frac{n}{H}\right)\exp\left(i\frac{2\pi n}{H}\right)\exp(-i2\pi k n/H) \\ &= \sum_{n=0}^{H-1} r_n \exp\left(i\frac{3\pi n}{2H}\right)\exp(-i2\pi k n/H). \end{aligned} \tag{A-8}$$

Therefore

$$W_{2k-1} = Re\left(\alpha_{H+1-k}\right), \quad 1 \le 2k-1 \le H. \tag{A-9}$$

Expanding (A-6) and (A-9) out in detail, there follows (for H even)

$$\begin{aligned}
W_1 &= Re(\alpha_H) = Re(\alpha_0) & W_2 &= Re(\alpha_1) \\
W_3 &= Re(\alpha_{H-1}) & W_4 &= Re(\alpha_2) \\
&\vdots & &\vdots \\
W_{H-3} &= Re\left(\alpha_{\frac{H}{2}+2}\right) & W_{H-2} &= Re\left(\alpha_{\frac{H}{2}-1}\right) \\
W_{H-1} &= Re\left(\alpha_{\frac{H}{2}+1}\right) & W_H &= Re\left(\alpha_{\frac{H}{2}}\right).
\end{aligned} \tag{A-10}$$

Thus the real parts of the H-point FFT in (A-5) for $0 \le k \le H-1$ serve to determine all the weights $\{w_k\}_1^H$.

APPENDIX B. PROGRAMS

The FFTs required in the following two subroutines are those presented in Refs. 6 and 7. The programs are written for H equal to a power of 2, but can be generalized via the technique of Appendix A.

```
      SUBROUTINE DOLPHS(H,DB,X,Y,C)
      DOUBLE PRECISION H21,Z0,A,T,C1,C2,T1,T2
      INTEGER H   C PROGRAM WRITTEN FOR H=2**INTEGER
      DIMENSION X(1),Y(1),C(1)
      I21=H*2-1
      H21=I21
      Z0=10.D0**(.05D0*ABS(DB))
      Z0=LOG(Z0+SQRT(Z0*Z0-1.D0))
      Z0=COSH(Z0/H21)
      A=1.5707963267948966190D0/H
      I1=H/2-1
      I2=H/4+2
      X(1)=.5D0*T(Z0)
      Y(1)=0.
      DO 1 K=1,I1
      C1=COS(A*K)
      C2=SIN(A*K)
      IF(MOD(K,4).NE.0) GO TO 2
      I3=K/4+1
      C(I3)=C1
      C(I2-I3)=C2
2     T1=T(Z0*C1)
      T2=T(Z0*C2)
      X(K+1)=T1*C1
      Y(K+1)=T1*C2
      X(H+1-K)=T2*C2
1     Y(H+1-K)=T2*C1
      C1=SQRT(.5D0)
      I3=H/8+1
      C(1)=1.
      C(I3)=C1
      C(I2-1)=0.
      T1=T(Z0*C1)
      X(I1+2)=T1*C1
      Y(I1+2)=X(I1+2)
```

```
      K=1.4427*LOG(H)+.5
      CALL MKLFFT(X,Y,C,K,-1)
      B=1./H
      I1=H-3
      DO 3 K=1,I1,2
      I2=(H+3-K)/2
      I3=(H+3+K)/2
      C(K)=B*X(I2)
3     C(K+1)=B*X(I3)
      C(H-1)=B*X(2)
      C(H)=B*X(1)
      RETURN
      FUNCTION T(X)
      DOUBLE PRECISION X
      IF(X.GT.1.DO) GO TO 1
      T=COS(H21*ACOS(X))
      RETURN
1     T=(X+SQRT(X*X-1.DO))**I21
      T=.5DO*(T+1.DO/T)
      RETURN
      END
```

```
      SUBROUTINE DOLPHD(H,DB,X,Y,C)
      DOUBLE PRECISION H21,Z0,A,T,C1,C2,T1,T2,X(1),Y(1),C(1)
      INTEGER H  @ PROGRAM WRITTEN FOR H=2**INTEGER
      I21=H*2-1
      H21=I21
      Z0=10.DO**(.05DO*ABS(DB))
      Z0=LOG(Z0+SQRT(Z0*Z0-1.DO))
      Z0=COSH(Z0/H21)
      A=1.5707963267948966190O/H
      I1=H/2-1
      I2=H/4+2
```

```
      X(1)=.5DO*T(ZO)
      Y(1)=0.DO
      DO 1 K=1,I1
      C1=COS(A*K)
      C2=SIN(A*K)
      IF(MOD(K,4).NE.0) GO TO 2
      I3=K/4+1
      C(I3)=C1
      C(I2-I3)=C2
2     T1=T(ZO*C1)
      T2=T(ZO*C2)
      X(K+1)=T1*C1
      Y(K+1)=T1*C2
      X(H+1-K)=T2*C2
1     Y(H+1-K)=T2*C1
      C1=SQRT(.5DO)
      I3=H/8+1
      C(1)=1.DO
      C(I3)=C1
      C(I2-1)=0.DO
      T1=T(ZO*C1)
      X(I1+2)=T1*C1
      Y(I1+2)=X(I1+2)
      K=1.4427*LOG(H)+.5
      CALL DPMFFT(X,Y,C,K,-1)
      A=1.DO/H
      I1=H-3
      DO 3 K=1,I1,2
      I2=(H+3-K)/2
      I3=(H+3+K)/2
      C(K)=A*X(I2)
      C(K+1)=A*X(I2)
      C(H-1)=A*X(2)
      C(H)=A*X(1)
      RETURN
      FUNCTION T(X)
      DOUBLE PRECISION X
      IF(X.GT.1.DO) GO TO 1
      T=COS(H21*ACOS(X))
      RETURN
1     T=(X+SQRT(X*X-1.DO))**I21
      T=.5DO*(T+1.DO/T)
      RETURN
      END
```